# Unified Nonlinear Analysis for Nonhomogeneous Anisotropic Beams with Closed Cross Sections

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**A unified methodology is presented for analysis of nonhomogeneous, anisotropic beams. Based on geometrically nonlinear, three-dimensional elasticity, and subject only to the restrictions that strain and local rotation are small compared to unity and that warping displacements (the deformation in and out of the cross-sectional plane) are small relative to cross-sectional dimensions, a two-dimensional analysis is derived that enables the determination of sectional elastic constants for the beam. Effects typical of open-section beams, such as restrained warping, are assumed to be negligible; thus, the analysis is well suited for beams with closed cross sections. From application of the finite element method, it is observed that the equations containing warping degrees of freedom are identical in form to those of an existing, purely linear analysis in the literature. There are two differences in the analysis as a whole, however: 1) the linear strain measures of the published analysis are replaced by certain generalized strain measures that are nonlinear functions of the displacement of the beam reference axis and rotation of the cross-sectional frame; and 2) the linear global equilibrium equations are replaced by exact, nonlinear, intrinsic ones. The structure of the governing equations tells us that the warping solutions can be affected by large deformation and that this could alter the incremental stiffness of the section. For a certain range of deformation, however, the elastic constants based on the reference state are adequate. As a result, it is shown that sectional constants derived from the published, linear analysis can be used in the present nonlinear, one-dimensional analysis governing the global deformation of the beam, which is based on the intrinsic equations for nonlinear beam behavior. The global behavior is determined by means of a mixed finite element analysis based on the weak form of all equilibrium, constitutive, and kinematical equations, including boundary conditions and continuity requirements. In spite of the simplicity of the approach, excellent correlation is obtained with published experimental results for both isotropic and anisotropic beams undergoing large deflections.**

## Introduction

WHEN a flexible structure has one dimension that is larger than the other two, it can often be treated as a beam, a one-dimensional structure. Many engineering structures can be idealized as beams, leading to much simpler equations than would be obtained if complete three-dimensional elasticity were used to model the structure. Although dimensional reduction processes can be extremely simple for homogeneous, isotropic, prismatic beams, and especially for restricted cases of deformation, they are far less tractable for composite beams undergoing arbitrary deformation. Specifically, difficulties arise in obtaining a one-dimensional strain energy function that is equivalent, at least in some sense, to a three-dimensional representation. For anisotropic beams, all possible deformations of the three-dimensional structure must be included in the formulation. This in turn suggests that it is necessary to remove the well-known restrictions that are typically imposed in a beam analysis.

St. Venant's solutions reveal well-known behavior for linear deformation of homogeneous, isotropic, prismatic beams:

1) Cross sections remain plane and normal to the beam axis during pure extensional deformation, but they contract in their own plane due to Poisson effects.

2) Similarly, cross sections remain plane and normal to the neutral axis during pure bending deformation; their shape is altered in their own plane, however, due to Poisson effects that are manifested in the so-called anticlastic curvature.

3) In pure torsion, the cross section does not remain plane (except for certain special shapes); its shape and size in its own plane, however, are preserved in this case.

4) Finally, in the case of a beam undergoing a constant shear force, which also implies a linearly varying bending moment (the so-called flexure problem), the shape of the cross section in its own plane is distorted due to the bending, and it also deforms out of its plane due to the shear.

In this paper, we refer to all such three-dimensional deformations of the cross section as "warping."

For homogeneous, isotropic, prismatic beams, there are thus six fundamental stiffnesses, one each for the six fundamental global deformations (extension and bending in the two principal directions, torsion and shear in the two principal directions). St. Venant's solutions allow for the determination of axial and bending stiffnesses in terms of simple integrals over the cross section; torsion and shear stiffnesses, however, depend on the out-of-plane warping functions. Closed-form solutions for torsional and shear warping functions exist for a few sections, but in general they must be determined by some sort of approximate method such as the finite-element method. It should be noted that the St. Venant solutions are frequently exploited for determination of beam section properties—even for beams that do not meet these narrow requirements.

For composite beams, much of this changes. In modern composite beams, there may be elastic couplings among all the global deformations. This means that instead of 6 fundamen-

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tal stiffnesses, there could be as many as 21 (a fully populated, symmetric $6 \times 6$ matrix). Furthermore, simple integrals over the cross section will not suffice to determine these elastic constants. The in- and out-of-plane warping deformations are coupled, and their effect on the stiffnesses may be much more significant. These complexities generally invalidate classical Kirchhoff-Clebsch and Euler-Bernoulli hypotheses and make the determination of the elastic constants (what is termed herein as "modeling") a much more difficult task. Linear St. Venant solutions for anisotropic beams with arbitrary cross section are summarized in Ref. 1.

For a review of the literature on the subject of modeling composite rotor blades as beams, see Ref. 2. Examples of such analyses include those of Berdichevsky,[3] Rehfield and Murthy,[4] Hong and Chopra,[5] Rehfield,[6] Bauchau,[7] Borri and Merlini,[8] Kosmatka,[9] Rehfield and Atılgan,[10] Lee and Stemple,[11] and Bauchau and Hong.[12] Most of the existing analyses invoke assumptions that restrict the model from being used for certain types of cross sections or deformation; also, most do not stem from the same theory as that which governs the global deformations. When both analyses are derivable from one common theory, we term that a "unified" approach.

Concerning a unified theory, Parker[13,14] found that the St. Venant solutions from linear beam theory play an important role in geometrically nonlinear analysis of beams. Berdichevsky,[3] however, appears to be the first in the literature to plainly state that "the geometrically nonlinear problem of the three-dimensional theory of elasticity for a beam can be split into a nonlinear one-dimensional problem and a linear two-dimensional problem." Parker[13,14] followed a formal asymptotic procedure, whereas Berdichevsky[3] followed his variational-asymptotic analysis.[15] Also, Berdichevsky[3] considered the general nonhomogeneous, anisotropic beam problem while Parker[13] considered only materials as general as monoclinic. However, both analyses put the contributions coming from the gradient of the warping along the beam axis into a higher approximation. Therefore, the flexure problem is only solved approximately by Parker,[14] whereas Berdichevsky[3] does not develop the flexure analysis at all. A later variational-asymptotic formulation[16] did consider the flexure problem; nevertheless, the authors treated only the isotropic case.

As a result of Refs. 13–16, we note that a consistent three-dimensional representation leads to a sort of decoupling of the modeling (a linear problem leading to the determination of the elastic constants for a cross section) from the global deformation analysis (which makes use of these constants and may be nonlinear).

Although they were apparently unaware of Berdichevsky's and Parker's work, Borri and Mantegazza[17] made use of this decoupling methodology as well. Their methods appear to be motivated by the work of Cosserat and Cosserat[18] and Ericksen and Truesdell.[19] These works require a constitutive law that connects the exact kinematics and kinetics of a one-dimensional continuum (i.e., regarding the beam as a solid generated by a two-dimensional section running along a curved line). Following Cosserat beam theory, Borri and Mantegazza[17] did not attempt to unify this methodology as two- and one-dimensional parts of a three-dimensional beam theory. Rather, they used a linear, two-dimensional finite element analysis,[20] implemented by Borri and his co-workers in a desktop computer program called Nonhomogeneous Anisotropic Beam Section Analysis (NABSA), to find the cross-sectional stiffness constants. It is the authors' opinion that Ref. 20/NABSA is the most general linear cross-sectional analysis/computer program to date. It correctly accounts for the flexure problem and is valid for general nonhomogeneous, anisotropic cross sections.

The purpose of this paper is to derive a unified theory governing both sectional and global deformation. To this end, the three-dimensional strain field is derived for arbitrary warping based on the kinematical relationships developed by Danielson and Hodges.[21] Next, the three-dimensional strain energy based on this strain field is used in the principle of virtual work to obtain a system of equations governing beam sectional and global deformation. We then show how these equations lead to a cross-sectional analysis of the form of Ref. 20. Thus, elastic constants calculated from NABSA are used in the present nonlinear, one-dimensional, global deformation analysis. A schematic of this unified process is shown in Fig. 1. For the global analysis, a weak form (i.e., a mixed variational method) based on exact intrinsic equilibrium and kinematical equations, specialized for the static case, is presented. Finally, correlations with experiments are given as a means of validation.

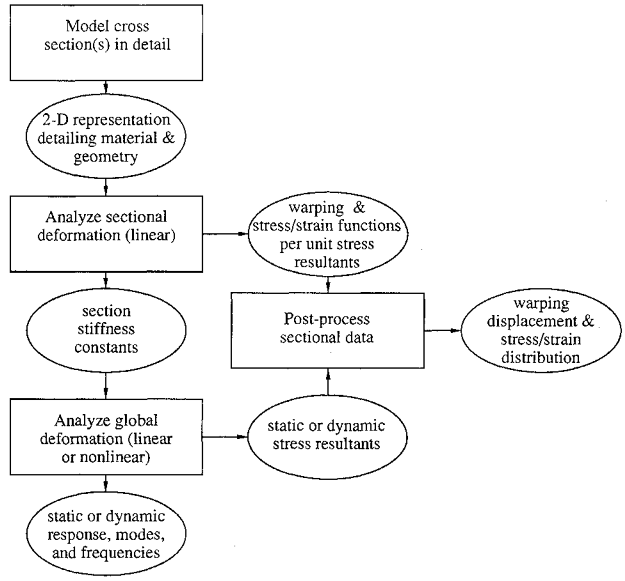


**Fig. 1 Schematic of a unified process for analyzing composite beams.**

## Kinematics

We are concerned here with deformations having small strains and small local rotations based on application of the general development of the strain field of a beam by Danielson and Hodges.[21] Thus, the deformation of the beam is close to a situation in which each cross section experiences a rigid-body motion. Each cross section remains nearly plane, although its orientation and displacement vary rather gradually along the beam reference axis, so that restrained warping ("edge zone") effects are not treated.

The distance along the beam reference axis is denoted by $x_1$. We now assume for simplicity that the beam is initially straight. However, this assumption is made strictly for illustrative purposes, since our general methodology does not require such an assumption. The position vector to an arbitrary point in the undeformed beam can be represented by

$$r^*(x_1,x_2,x_3) = x_i \boldsymbol{b}_i \tag{1}$$

where the undeformed beam reference triad $\boldsymbol{b}_i$ has a fixed orientation in space with $\boldsymbol{b}_1$ parallel to the beam axial coordinate $x_1$ and $\boldsymbol{b}_\alpha$ parallel to Cartesian coordinate lines $x_\alpha$ in the cross-section plane. (Note that Roman indices vary from 1 to 3, while Greek indices vary from 2 to 3; repeated indices are summed over their range.) The covariant base vectors for the undeformed state are simply

$$\boldsymbol{g}_i \triangleq \frac{\partial r^*}{\partial x_i} = \boldsymbol{b}_i \tag{2}$$

The contravariant base vectors are

$$\boldsymbol{g}^i \triangleq \tfrac{1}{2} e_{ijk} \boldsymbol{g}_j \times \boldsymbol{g}_k = \boldsymbol{b}_i \tag{3}$$

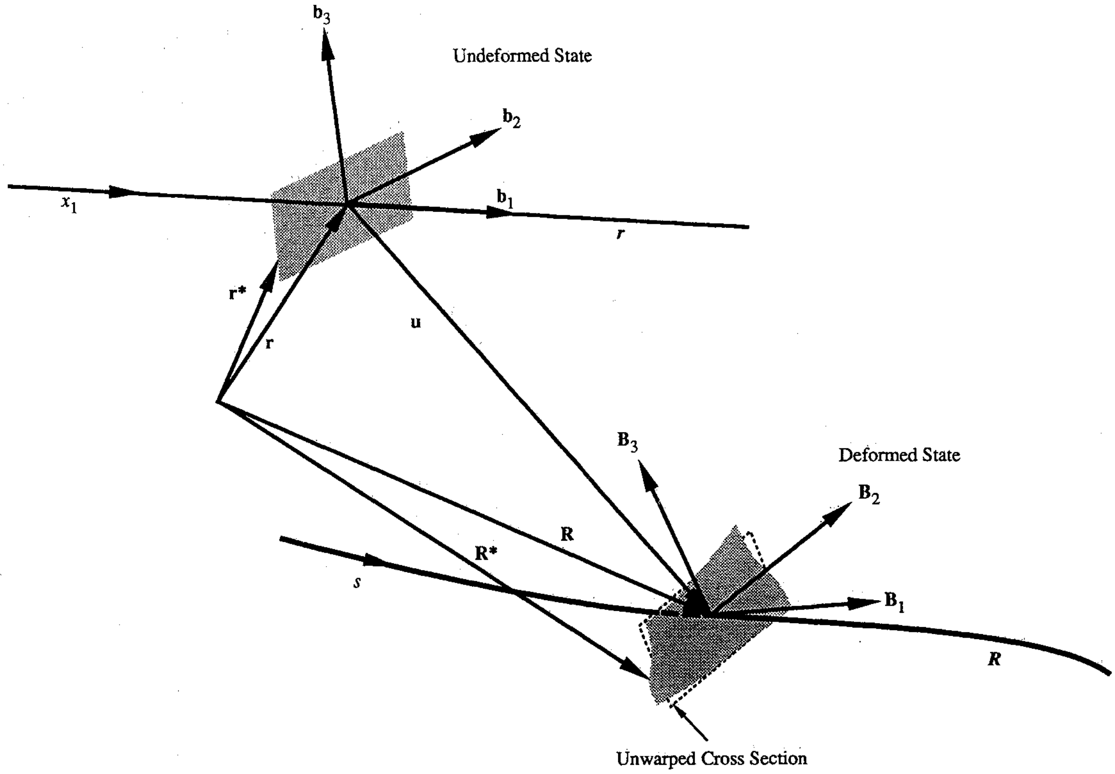


**Fig. 2 Geometry of the beam deformation.**

The position vector to the same arbitrary point in the deformed beam can be represented by

$$R^*(x_1,x_2,x_3) = x_1 b_1 + u(x_1) + x_\alpha B_\alpha(x_1) + w_i(x_1,x_2,x_3) B_i(x_1) \tag{4}$$

where $u = u_i b_i$ is the displacement vector of points on the reference line and $w_i$ is the general warping displacement of an arbitrary point in the cross section, consisting of components both in and out of the plane of the cross section (Fig. 2). For a beam undergoing small strain, the unit vectors $B_\alpha$ can always be chosen so that $w_i$ remains small. Now if $B_1 = B_2 \times B_3$, this will mean that $B_1$ is not necessarily tangent to the curved line of material points in the deformed beam that were on the reference axis in the undeformed beam. The orthogonal triad $B_i$ is referred to hereafter as the deformed beam basis; these vectors are related to those associated with the undeformed beam by

$$B_i(x_1) = C_{ij}(x_1) b_j = C(x_1) \cdot b_i \tag{5}$$

where $C$ is the matrix of direction cosines whose elements are components of the rotation tensor $C$. The covariant base vectors for the deformed state are defined as

$$G_i \triangleq \frac{\partial R^*}{\partial x_i} \tag{6}$$

Introducing $(\ )'$ to denote the partial derivative with respect to $x_1$ and $(\ )_{,\alpha}$ to denote the partial derivative with respect to $x_\alpha$, one can express these vectors as

$$G_1 = (1+\gamma_{11}) B_1 + 2\gamma_{1\alpha} B_\alpha + x_\alpha \kappa_i B_i \times B_\alpha + w_i' B_i + w_i \kappa_j B_j \times B_i \tag{7a}$$

$$G_\alpha = B_\alpha + w_{i,\alpha} B_i \tag{7b}$$

where $\gamma_{11}$ is the extensional strain, $2\gamma_{12}$ and $2\gamma_{13}$ are the shear strains at the sectional reference point (that point in the section coincident with the reference line), $\kappa_1$ is the elastic twist per unit length, and $\kappa_2$ and $\kappa_3$ are components of the bending curvature. These can be conveniently arranged in matrix form

$$\gamma \triangleq \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \qquad \kappa \triangleq \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \qquad \epsilon \triangleq \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \tag{8}$$

Introducing $u = [u_1\ u_2\ u_3]^T$, one can express $\gamma$ and $\kappa$ as

$$\gamma = C(u' + e_1) - e_1 \tag{9a}$$

$$\tilde{\kappa} = -C' C^T \tag{9b}$$

where $(\tilde{\ })_{ij} = -e_{ijk}(\ )_k$, $e_{ijk}$ is the permutation symbol, and $e_1 = [1\ 0\ 0]^T$. The force strain $\gamma$ is so designated because it is conjugate to the actual section force as shown in Ref. 22. Similarly, $\kappa$ is called the moment strain because it is conjugate to the actual section moment.

We denote the Jaumann-Biot-Cauchy strain field components by $\Gamma^*$, a $3\times 3$ symmetric matrix. For small strain and small local rotation, Ref. 21 shows that

$$\Gamma^* = \frac{\chi + \chi^T}{2} - I \tag{10}$$

where $I$ is the $3\times 3$ identity matrix and $\chi$ is the matrix of deformation gradient components in mixed bases given by

$$\chi_{ij} = B_i \cdot G_k g^k \cdot b_j = B_i \cdot G_j \tag{11}$$

The components $\Gamma_{ij}$ of the strain field of the matrix $\Gamma^*$ are then found to be

$$\Gamma_{11} = \gamma_{11} + w_1' + \tilde{\kappa}_{1\alpha}(x_\alpha + \underline{w_\alpha}) \tag{12a}$$

$$2\Gamma_{1\alpha} = 2\gamma_{1\alpha} + w_{1,\alpha} + w_\alpha' + \tilde{\kappa}_{\alpha\beta} x_\beta + \underline{\tilde{\kappa}_{\alpha j} w_j} \tag{12b}$$

$$2\Gamma_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha} \tag{12c}$$

If we further suppose that warping displacements are very small relative to the cross-sectional dimensions so that $\max(w_\alpha) \ll \max(x_\alpha)$ and that $\kappa_\alpha \max(w_1) \ll \kappa_1 \max(x_\alpha)$, the underlined terms in the strain field can be neglected. (Neither these assumptions nor that of small local rotation would be appropriate for thin-walled, open-section beams.) Introducing column matrices $w$ and $\xi$ containing, respectively, the components $w_i$ and $[0\ x_2\ x_3]^T$, and operator matrices

$$\partial_1 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \end{bmatrix} \qquad \partial_2 \triangleq \begin{bmatrix} 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \tag{13}$$

one can express the strain field as a $6\times 1$ column matrix $\Gamma = [\Gamma_{11}\ 2\Gamma_{12}\ 2\Gamma_{13}\ \Gamma_{22}\ 2\Gamma_{23}\ \Gamma_{33}]^T$ so that

$$\Gamma = \begin{bmatrix} I & -\tilde{\xi} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} w' + \begin{bmatrix} \partial_1 \\ \partial_2 \end{bmatrix} w$$

$$= X\epsilon + Yw' + \partial w \tag{14}$$

The form of the strain field is of great importance, because it is now linear in $\gamma$, $\kappa$, and $w$ and its derivatives. This gives some hope that, with proper constraints, the warping $w$ could be eliminated in terms of $\gamma$ and $\kappa$ (or their conjugates) by using linear differential equations whose domain is the cross-sectional plane.

To obtain the contribution of the internal forces to the equations of equilibrium from the strain energy, it is necessary to take variations of the strain measures (i.e., $\delta\gamma$ and $\delta\kappa$). Furthermore, since we want to obtain the final equations of equilibrium in intrinsic form (a form that is independent of the variables used to express the displacement and rotation of the beam reference line and cross section), we must take the variations with this objective in mind. The relations between the variations of these strain measures and the virtual displacement and virtual rotation measures are sometimes referred to as transpositional or transitivity relations. The derivation of these relations below is an abbreviated and specialized version of that given in Ref. 23.

First, the variation of $\kappa$ is obtained from Eq. (9b)

$$\delta\tilde{\kappa} = -\delta C' C^T - C'\delta C^T \tag{15}$$

To further develop this variation, in accordance with standard treatments,[24] the virtual rotation must be introduced. It can be found by replacing ( )′ with $\delta$( ) in Eq. (9b) so that

$$\widetilde{\overline{\delta\psi}} = -\delta C C^T \tag{16}$$

In this expression, the overbar indicates that $\overline{\delta\psi}$ is not the variation of a function, as can be observed. Differentiation of Eq. (16) with respect to $x_1$ leads to an expression that can be used to eliminate the $\delta C'$ from Eq. (15). One can then solve Eq. (9b) for $C'$ and Eq. (16) for $\delta C$

$$C' = -\tilde{\kappa} C \tag{17a}$$

$$\delta C = -\widetilde{\overline{\delta\psi}}\, C \tag{17b}$$

and then use Eqs. (17) to eliminate $C'$ and $\delta C$ from Eq. (15). Easily derived identities associated with the tilde notation ($\tilde{a}b = -\tilde{b}a$ and $\widetilde{\tilde{a}b}\,\tilde{a}\tilde{b} = \tilde{a}\tilde{b} - \tilde{b}\tilde{a}$) can be used to finally obtain

$$\delta\kappa = \overline{\delta\psi}' + \tilde{\kappa}\overline{\delta\psi} \tag{18}$$

The variation of the force strain can be carried out in a similar fashion. From Eq. (9a), the variation of $\gamma$ is

$$\delta\gamma = \delta C(e_1 + u') + C\delta u' \tag{19}$$

Using Eq. (9a) again, we can write the first term on the right-hand side as

$$\delta C(e_1 + u') = \delta C C^T(e_1 + \gamma) \tag{20}$$

To obtain a compatible virtual displacement, we replace the ( )′ with $\delta$( ) in Eq. (9a) and ignore the other terms yielding

$$\overline{\delta q} = C\delta u \tag{21}$$

Differentiating Eq. (21), one can obtain an expression for the second term on the right-hand side of Eq. (19), $C\delta u'$, that is independent of $u$

$$C\delta u' = \overline{\delta q}' - C'C^T\overline{\delta q} \tag{22}$$

From Eqs. (17) and (19–22) one can now write

$$\delta\gamma = \overline{\delta q}' + \tilde{\kappa}\overline{\delta q} + (\tilde{e}_1 + \tilde{\gamma})\overline{\delta\psi} \tag{23}$$

Equations (18) and (23) are the relations needed to express the variation of the strain energy in terms of intrinsic quantities only. These equations agree with similar relations derived in Ref. 25 in a completely different manner. They can be expressed in a $6\times 1$ column matrix form by rearranging Eqs. (23) and (18) to obtain

$$\delta\epsilon = \begin{Bmatrix} \overline{\delta q} \\ \overline{\delta\psi} \end{Bmatrix}' + \begin{bmatrix} \tilde{\kappa} & \tilde{e}_1 + \tilde{\gamma} \\ 0 & \tilde{\kappa} \end{bmatrix} \begin{Bmatrix} \overline{\delta q} \\ \overline{\delta\psi} \end{Bmatrix} \tag{24}$$

Finally, to express the virtual work of tractions distributed over the beam cross-sectional face, let us introduce $\overline{\delta s}$, a column matrix of the measure numbers of the virtual displacement in the deformed beam basis. Furthermore, let us resolve $\overline{\delta s}$ into a part that does not strain the cross section and a residual part representing the variation of cross-sectional warping $\delta w$. Assuming all elements of $\overline{\delta\psi}$ are of the same order and that $\max(w_i) \ll \max(x_\alpha)$, one can obtain

$$\overline{\delta s} = \delta w + \overline{\delta q} - \tilde{\xi}\overline{\delta\psi} \tag{25}$$

The relations developed in this section will be used next to obtain the final form of the unified analysis.

## Three-Dimensional Analysis

In this section we consider an infinitesimal "slice" of a loaded beam as a starting point for developing the cross-sectional and global deformation analyses. To proceed, we first must represent the stress field in the beam. For this, we choose the Jaumann stress measure $\boldsymbol{Z} = \boldsymbol{b}_i Z_{ij} \boldsymbol{b}_j$. This stress measure was used by Fraeijs de Veubeke,[26] and its conjugacy with $\Gamma$, viz.,

$$Z = \left(\frac{\partial \mathcal{W}}{\partial \Gamma}\right)^T = \mathfrak{D}\Gamma \tag{26}$$

was derived by Ogden[27] and Atluri.[28] Here $\mathcal{W}$ is the strain energy and $Z$ is in a $6\times 1$ column matrix form (similar to $\Gamma$)

$$Z \triangleq \begin{Bmatrix} Z_s \\ Z_t \end{Bmatrix} \qquad Z_s \triangleq \begin{Bmatrix} Z_{11} \\ Z_{12} \\ Z_{13} \end{Bmatrix} \qquad Z_t \triangleq \begin{Bmatrix} Z_{22} \\ Z_{23} \\ Z_{33} \end{Bmatrix} \tag{27}$$

The notation $Z_s$ represents the stresses acting on the surface normal to $x_1$ (i.e., the two-dimensional surface of material points that occupied a reference cross section of the undeformed beam), while $Z_t$ are the remaining stresses. The $Z_t$ is normally set equal to zero in beam theory (to obtain the appropriate stiffness constants for extension, shear, bending, and torsion), but for anisotropic beam theory this will not suffice. The differential force acting on an element of this surface is given by

$$b_1 \cdot Z \cdot C^T \, dx_2 \, dx_3 = Z_{1i} B_i \, dx_2 \, dx_3 \tag{28}$$

Let us consider now a part of the beam between two cross sections having an infinitesimally small distance between them. One can write the principle of virtual work for this "beam slice" as

$$\left( \iint_{\mathfrak{a}} \overline{\delta s}^T Z_s \, dx_2 \, dx_3 \right)' + \overline{\delta q}^T f + \overline{\delta \psi}^T m = \iint_{\mathfrak{a}} \delta \Gamma^T Z \, dx_2 \, dx_3 \tag{29}$$

where $f$ and $m$ are column matrices containing the measure numbers of the applied force at the sectional reference point and the applied moment about the sectional reference point, both per unit length and both in the deformed beam basis ($B_i$). Thus, $f$ contains measure numbers of the resultant of all body forces and exterior surface tractions, and $m$ contains measure numbers of the moment of all body forces and exterior surface tractions about the sectional reference point. The contribution of $\delta w$ to the virtual work of the applied surface tractions and body forces is ignored in the present development. (Asymptotical treatments[3,13] of this problem typically ignore such contributions or place them in higher approximations.)

If we now plug Eq. (25) into the left-hand side of Eq. (29), the result is

$$\overline{\delta q}^T f + \overline{\delta \psi}^T m + (\overline{\delta q}^T F)' + (\overline{\delta \psi}^T M)' = \iint_{\mathfrak{a}} \left[ \delta \Gamma^T Z - (\delta w^T Z_s)' \right] dx_2 \, dx_3 \tag{30}$$

where $F$ and $M$ contain measure numbers of the cross-sectional force and moment stress resultants in the deformed beam basis and are given by

$$Q \triangleq \begin{Bmatrix} F \\ M \end{Bmatrix} \qquad F \triangleq \iint_{\mathfrak{a}} Z_s \, dx_2 \, dx_3 = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$M \triangleq \iint_{\mathfrak{a}} \tilde{\xi} Z_s \, dx_2 \, dx_3 = \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{31}$$

Here $F_1$ is the axial force, $F_2$ and $F_3$ are the shear forces, $M_1$ is torsional moment, and $M_2$ and $M_3$ are the bending moments. The right-hand side of Eq. (30) is independent of $\overline{\delta q}$ and $\overline{\delta \psi}$. Thus, after using the transpositional relations of Eq. (24) to eliminate $\overline{\delta q}'$ and $\overline{\delta \psi}'$, the left-hand side will yield the well-known beam intrinsic equations[29,25]

$$F' + \tilde{\kappa} F + f = 0 \tag{32a}$$

$$M' + (\tilde{e}_1 + \tilde{\gamma}) F + \tilde{\kappa} M + m = 0 \tag{32b}$$

or

$$Q' + \begin{bmatrix} \tilde{\kappa} & 0 \\ \tilde{e}_1 + \tilde{\gamma} & \tilde{\kappa} \end{bmatrix} Q + \begin{Bmatrix} f \\ m \end{Bmatrix} = 0 \tag{33}$$

which has a form similar to that of Eq. (24).

Now, to deal with the remaining part of Eq. (30), the cross-sectional warping is discretized as $w(x_1, x_2, x_3) = S(x_2, x_3) \cdot W(x_1)$, $S$ being a $3 \times n$ matrix of appropriate two-dimensional finite-element shape functions, $n$ being the number of nodal displacements $W$ over the cross section. Finally, using Eq. (14) in the right-hand side of Eq. (29), and Eqs. (33) and (24) in the left-hand side, the two-dimensional principle of virtual work leads to a system of finite-element equations governing the warping

$$JW'' - HW' - EW - R\epsilon + L\epsilon' = 0 \tag{34a}$$

$$L^T W' + R^T W + K_0 \epsilon = Q \tag{34b}$$

where the redundancy of 6 degrees of freedom present in $W$ has been appropriately removed as discussed by Ref. 20. The matrices $K_0$, $E$, $R$, $H$, $L$, and $J$ are defined as

$$K_0 = \iint_{\mathfrak{a}} X^T \mathfrak{D} X \, dx_2 \, dx_3 \qquad E = \iint_{\mathfrak{a}} (\partial S)^T \mathfrak{D} \partial S \, dx_2 \, dx_3$$

$$R = \iint_{\mathfrak{a}} (\partial S)^T \mathfrak{D} X \, dx_2 \, dx_3$$

$$H = \iint_{\mathfrak{a}} \left[ (\partial S)^T \mathfrak{D} Y S - S^T Y^T \mathfrak{D} \partial S \right] dx_2 \, dx_3$$

$$L = \iint_{\mathfrak{a}} S^T Y^T \mathfrak{D} X \, dx_2 \, dx_3 \qquad J = \iint_{\mathfrak{a}} S^T Y^T \mathfrak{D} Y S \, dx_2 \, dx_3 \tag{35}$$

all of which depend on the cross-sectional geometry and material. Here, $J$, $H$, and $E$ are $n \times n$ matrices, $R$ and $L$ are $n \times 6$, and $K_0$ is $6 \times 6$.

Thus, Eqs. (34) and (33) are the Euler-Lagrange equations for Eq. (29). There are $12 + n$ equations and $12 + n$ unknowns $\epsilon$, $Q$, and $W$, all functions of $x_1$ only. To solve these equations as they are, however, would require carrying a large number of unknowns since $n$ may have to be quite large to model a general cross-sectional geometry and material distribution. An alternative is clearly desirable from the point of view of computational efficiency.

## Cross-Sectional Analysis

Rather than carrying this large number of unknowns, it is reasonable to carry no more than 12 ($Q$ and $\epsilon$) in the global deformation analysis; thus we must find a convenient and accurate way of eliminating $W$. This is the usual philosophy of beam theory, although this point may not be obvious from the above development. Since we are interested in finding the section properties of the beam, and we understand that the section properties of most beams are not strongly dependent on the deformation, we look for ways to eliminate the warping that will yield section constants that are independent of deformation. We now develop several special cases that should be helpful to the reader both in making identification with well-known theories and in finding a practical means of solving beam equations.

### Direct Elimination of Warping

A simplistic approximation is to ignore the warping altogether. Equation (34a) cannot be used if we set $W = \delta W = 0$. From Eq. (34b) we obtain $Q = K_0 \epsilon$. In other words, $K_0$ would be the sectional stiffness matrix for the case in which there is no warping. This represents what we refer to as the *uniaxial strain* condition ($\Gamma_{\alpha\beta} = 0$) in which the section is not allowed to undergo any in-plane warping (neither Poisson contraction nor anticlastic deformation). In the isotropic case, this means that extension and bending stiffnesses are too large by a factor of $(1-\nu)/(1+\nu)(1-2\nu)$. Furthermore, because there is no out-of-plane warping, the torsional stiffness is grossly overestimated as well. In fact, the elements of $K_0$ are upper bounds for the sectional stiffness constants. Equation (33) then governs the global deformation.

Another related possibility is to enforce the *uniaxial stress* condition ($Z_{\alpha\beta}=0$). This condition can be shown to give correct values of extension, bending, and torsional stiffness constants for the isotropic case, but one cannot expect a priori to obtain accurate stiffnesses for composite beams this way. We will not develop the uniaxial stress solution in detail here.

When the body forces and the transverse stresses on the lateral (external) surface of the beam both vanish, we obtain the conditions appropriate for solving the St. Venant cross-sectional problem. In this case, $f = m = 0$ and Eq. (29) can be shown to be a weak form of the equilibrium equations and the natural boundary conditions for the St. Venant problem.[20] Furthermore, Eqs. (34) and (33) with $f = m = 0$ govern a geometrically nonlinear form of the St. Venant problem. These equations are identical to those of Ref. 20 upon which the program NABSA, mentioned earlier, is based except that 1) the present approach is developed in terms of geometrically nonlinear strain measures $\gamma$ and $\kappa$, and 2) the beam equilibrium equation, Eq. (33), is obtained here in its nonliner intrinsic form. We are interested in seeing how far one can go in solving the St. Venant problem in the geometrically nonlinear case.

One special case can be dealt with immediately; that is, to treat extension, bending, and torsional (EBT) solutions individually for homogeneous isotropic beams. In these cases, the warping and strain do not vary along the beam, and one can set all $(\ )'$ quantities equal to zero in Eqs. (34). (It should be noted that there is only a 4 degree-of-freedom redundancy with transverse shear deformation set equal to zero.) This leaves only two sets of algebraic equations from which $W$ can be eliminated, resulting in a strain energy per unit length

$$U \triangleq \frac{1}{2}\iint_{\alpha} \Gamma^T \mathfrak{D} \Gamma \, dx_2 \, dx_3 = \frac{1}{2}\begin{Bmatrix}\gamma_{11}\\ \kappa\end{Bmatrix}^T K_1 \begin{Bmatrix}\gamma_{11}\\ \kappa\end{Bmatrix} \quad \text{(EBT, Isotropic Case)} \tag{36}$$

where the sectional stiffness matrix is

$$K_1 = K_0 - R^T E^{-1} R \tag{37}$$

It can be verified that $K_1$, as more elements are taken over the section, converges to the classical stiffnesses for EBT-type deformations of isotropic beams. Note that $K_0$ and $K_1$ are $4\times4$ in Eqs. (36) and (37) because shear deformation has been set to zero.

It is necessary to solve the flexure problem to determine stiffness constants related to shear. As pointed out by several investigators (e.g., see Refs. 14, 30, and 31), the flexure problem in its geometrically nonlinear form is quite different from EBT problems, and it may not even be possible to solve it in this framework unless some sort of linearization process is employed. Thus, it is preferable to proceed on the basis of some approximations. Of the two types of approximations to be discussed here, the first involves turning the problem into a linear one in $Q$ and $\epsilon$, and the other involves asymptotic methods.

#### Quasilinear and Asymptotic Methods

Equations (34) are a linear system of ordinary differential equations in $\epsilon$ and $Q$, while Eq. (33), on the other hand, is nonlinear. For global deformations that are small, one can pursue a solution of Eqs. (34) coupled with a form of Eq. (33) that is linearized about the reference (undeformed and unstressed) state, again with $f = m = 0$ as above. Now this quasilinear form of the governing equations is exactly of the form used in NABSA (although the strain measures are nonlinear functions of the displacement and rotation—thus the term quasilinear). Therefore, solutions from NABSA should be reasonable approximations for obtaining the sectional elastic constants for use in nonlinear elastic deformation of slender, closed-section composite beams, as long as $\gamma$ and $\kappa$ are not too large. The structure of Eqs. (34) and (33) tells us, however, that the warping solutions could be affected by large deformation, and this could alter the incremental stiffness of the section. Just how large the global deformation measures would have to become to render the elastic constants based on the reference state inaccurate has not been investigated. In Ref. 17, the authors assumed the constants from the reference state to be adequate for nonlinear deformation without stating the limitations involved. This subject deserves further investigation as does the question of how reasonable NABSA approximations are for wider classes of nonlinear deformation of composite beams.

Consider the homogeneous, linear form of Eq. (33)

$$Q' + \begin{bmatrix} 0 & 0 \\ \widetilde{e_1} & 0 \end{bmatrix} Q = 0 \tag{38}$$

This along with Eqs. (34) make up what we call the quasilinear form of the cross-sectional problem. For EBT problems, we have $F_\alpha = \gamma_{1\alpha} = 0$. If the former is used in Eq. (38), it is easily seen that $Q' = 0$. Now it is possible to argue in the same manner as done in Ref. 20 that $W' = \epsilon' = 0$ as well. This leads immediately to a matrix of elastic sectional constants for generally nonhomogeneous, anisotropic materials of the form $K_1$ in Eq. (37), which was derived before based on the geometrically nonlinear equations. Thus, the quasilinear approach does yield a tractable result for EBT problems, and this result for the stiffness matrix is shown in Ref. 3 to be asymptotically correct for the first approximation (which corresponds to a geometrically nonlinear, anisotropic beam theory without transverse shear deformation).

If we now deal with shear deformation, as stated above, it is not possible to eliminate the warping associated with the flexure solution from the geometrically nonlinear equations without resorting to asymptotical considerations. For the quasilinear case, however, the equations governing the flexure problem are the same as those of Ref. 20, and thus NABSA gives the exact solution for the quasilinear equations. It is currently under investigation whether that solution is asymptotically correct for the second approximation (which corresponds to a geometrically nonlinear, anisotropic beam theory including transverse shear deformation). This theory would be analogous to that developed in Ref. 16 for the isotropic case.

Once the cross-sectional equations are solved, the resulting matrix of sectional elastic constants can be used to create a one-dimensional strain energy that depends only on the global deformation parameters $\gamma$ and $\kappa$. This means that there exists a strain energy density per unit length $U(\gamma,\kappa)$ such that

$$\left(\frac{\partial U}{\partial \gamma}\right)^T \triangleq F \qquad \left(\frac{\partial U}{\partial \kappa}\right)^T \triangleq M \tag{39}$$

and the elastic law can be put in a more explicit form such that

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \tag{40}$$

(The reader must not confuse these elastic constants with those of lamination theory. There is no relation!) This form of strain energy leads to the intrinsic nonlinear equilibrium equations as shown in Ref. 22. Therefore, the elastic constants for use in a unified nonlinear analysis of anisotropic beams may be based upon Ref. 20 and NABSA. This methodology was also followed by Borri and Mantegazza,[17] but without the unified framework shown earlier.

## Global Deformation Analysis

As a basis for a numerical method, a variational principle often plays an important role. In large displacement problems, so far, the principle of stationary potential energy has been more widely used, for instance by Borri and Mantegazza.[17] Pure displacement formulations, however, have now been abandoned by many researchers of finite element formulations, and general mixed variational formulations are receiving wide attention (see, for exmaple, Iura and Atluri[32]). Here we will derive a general mixed variational principle for the static behavior of finitely deformed beams. It should be noted that the present methodology is not limited to statics; the global equations of motion derived in Ref. 23 could also be used to analyze dynamic behavior in conjunction with NABSA. Indeed, linear free-vibration characteristics were analyzed using this method in Ref. 33.

A one-dimensional, geometrically nonlinear, boundary value problem for straight beams can be stated in matrix form as follows

$$F' + \tilde{\kappa}F + f = 0 \tag{41a}$$

$$M' + (\tilde{e}_1 + \tilde{\gamma})F + \tilde{\kappa}M + m = 0 \tag{41b}$$

$$A\gamma + B\kappa - F = 0 \tag{41c}$$

$$B^T\gamma + D\kappa - M = 0 \tag{41d}$$

$$u + e_1 - C^T(e_1 + \gamma) = 0 \tag{41e}$$

$$\theta' - \left(I + \frac{\tilde{\theta}}{2} + \frac{\theta\theta^T}{4}\right)\kappa = 0 \tag{41f}$$

where

$$C \triangleq \frac{(1 - \frac{1}{4}\theta^T\theta)I - \tilde{\theta} + \frac{1}{2}\theta\theta^T}{1 + \frac{1}{4}\theta^T\theta} \tag{42}$$

and where $\theta$ is the column matrix whose elements are the Rodrigues parameters $\theta_i$. Equations (41a) and (41b) are the force and moment equilibrium equations; Eqs. (41c) and (41d) are the constitutive relations, and Eqs. (41e) and (41f) are inverted forms of the kinematical relations derived in Ref. 25.

The associated boundary conditions are written as

$$F = \hat{F}, \qquad M = \hat{M} \qquad \text{on} \quad S_Z \tag{43}$$

$$u = \hat{u}, \qquad \theta = \hat{\theta} \qquad \text{on} \quad S_u \tag{44}$$

where $\hat{F}$ and $\hat{M}$ denote the prescribed values on the traction boundary $S_Z$ whereas $\hat{u}$ and $\hat{\theta}$ denote the prescribed values on the displacement boundary $S_u$.

Multiplying these field equations by the appropriate test functions and integrating by parts, one obtains the following combined weak form

$$\begin{aligned}\overline{\delta\pi} = \int_0^\ell \Big\{ & \delta\gamma^T(A\gamma + B\kappa - F) + \delta\kappa^T(B^T\gamma + D\kappa - M) \\ & + \overline{\delta F}^T[e_1 - C^T(e_1+\gamma)] - \overline{\delta F}'^T u - \overline{\delta M}^T\left(I + \frac{\tilde{\theta}}{2} + \frac{\theta\theta^T}{4}\right)\kappa \\ & - \overline{\delta M}'^T\theta + \left[\overline{\delta q}'^T - \overline{\delta q}^T\tilde{\kappa} - \overline{\delta\psi}^T(\tilde{e}_1 + \tilde{\gamma})\right]F + \overline{\delta q}^T f \\ & + \left(\overline{\delta\psi}'^T - \overline{\delta\psi}^T\tilde{\kappa}\right)M + \overline{\delta\psi}^T m \Big\}\, dx_1 \\ & + \left(\overline{\delta F}^T\hat{u} + \overline{\delta M}^T\hat{\theta} - \overline{\delta q}^T\hat{F} - \overline{\delta\psi}^T\hat{M}\right)\Big|_0^\ell = 0\end{aligned} \tag{45}$$

where

$$\overline{\delta F}^T = \delta F^T C \tag{46a}$$

$$\overline{\delta M}^T = \frac{\delta M^T(I - \frac{1}{2}\tilde{\theta})}{1 + \frac{1}{4}\theta^T\theta} \tag{46b}$$

This formulation is called the "weakest possible form" because all the one-dimensional field equations, namely, equilibrium equations, constitutive law, strain-displacement relationships, and boundary conditions are represented in their most basic form without differentiation of any field variables with respect to the axial coordinate.[34] An extension of this form including dynamics is derived in Ref. 23 from Hamilton's principle. Although this mixed weak approach creates a system with a large number of unknowns, the matrices involved are very sparse. This sparsity is significant since sparse matrix routines can be used to improve the formulation's efficiency. In addition, this approach allows the calculation of all of the unknowns to the same order of accuracy. For example, one need not differentiate displacements or rotations to calculate the generalized strains or the stress resultants. The advantage of using this weakest possible form as a basis for finite element formulation is that one can choose any class of shape function having the lowest possible continuity requirements. This allows one to perform the element level integrations by inspection. Not needing to use numerical quadrature saves additional computer time.

The perturbed form can be discretized by assuming that the beam is broken up into $N$ elements with nodes between elements numbered from 1 and the root to $N+1$ at the tip. Because of the weakness of the formulation, all quantities that appear in terms that are differentiated can be represented as piecewise linear, and all remaining terms piecewise constant. Thus, the virtual displacements, rotation, forces, and moments become, in the $i$th element

$$\overline{\delta q} = \overline{\delta q}_{(i)}(1 - \zeta) + \overline{\delta q}_{(i+1)}\zeta \tag{47a}$$

$$\overline{\delta\psi} = \overline{\delta\psi}_{(i)}(1 - \zeta) + \overline{\delta\psi}_{(i+1)}\zeta \tag{47b}$$

$$\overline{\delta F} = \overline{\delta F}_{(i)}(1 - \zeta) + \overline{\delta F}_{(i+1)}\zeta \tag{47c}$$

$$\overline{\delta M} = \overline{\delta M}_{(i)}(1 - \zeta) + \overline{\delta M}_{(i+1)}\zeta \tag{47d}$$

where the subscripts refer to node numbers along the beam (enclosed in parentheses to avoid confusion with the index of the column matrices) and $\zeta$ is a local element axial coordinate so that $0 \le \zeta \le 1$.

The unknowns corresponding to these quantities then are assumed to be piecewise constant within the element $(0 < \zeta < 1)$

$$u = \bar{u}_{(i)} \tag{48a}$$

$$\theta = \bar{\theta}_{(i)} \tag{48b}$$

$$F = \bar{F}_{(i)} \tag{48c}$$

$$M = \bar{M}_{(i)} \tag{48d}$$

but with distinct discrete values at the ends of the element (the nodal values)

$$u = \hat{u}_{(i)} \quad (\zeta = 0), \qquad u = \hat{u}_{(i+1)} \quad (\zeta = 1) \tag{49a}$$

$$\theta = \hat{\theta}_{(i)} \quad (\zeta = 0), \qquad \theta = \hat{\theta}_{(i+1)} \quad (\zeta = 1) \tag{49b}$$

$$F = \hat{F}_{(i)} \quad (\zeta = 0), \qquad F = \hat{F}_{(i+1)} \quad (\zeta = 1) \tag{49c}$$

$$M = \hat{M}_{(i)} \quad (\zeta = 0), \qquad M = \hat{M}_{(i+1)} \quad (\zeta = 1) \tag{49d}$$

In the absence of discrete applied forces and moments in the interior of the beam and displacement and rotational constraints within the beam, all the nodal variables drop out of the resulting algebraic equations during assembly except those at the extremities of the beam. For example, for a cantilever beam, those nodal values that are known at the extremities are $\hat{u}_{(1)} = \hat{\theta}_{(1)} = \hat{F}_{(N+1)} = \hat{M}_{(N+1)} = 0$. Equations governing $\hat{u}_{(N+1)}$, $\hat{\theta}_{(N+1)}$, $\hat{F}_{(1)}$, and $\hat{M}_{(1)}$, can be decoupled from the other equations and solved in a postprocessing operation to obtain these and all other nodal variables.

The remaining variables and their variations are assumed to be constants within the elements so that

$$\gamma = \overline{\gamma}_i, \qquad \delta\gamma = \overline{\delta\gamma}_i \tag{50a}$$

$$\kappa = \overline{\kappa}_i, \qquad \delta\kappa = \overline{\delta\kappa}_i \tag{50b}$$

This discretization may seem crude, but it is sufficient for this mixed, intrinsic formulation. In addition, these "crude" shape functions allow Eq. (45) to be integrated in closed form.

We have used equally spaced elements and assumed that properties of the beam are constant in the spanwise direction. These simplifications, of course, are not restrictions of the general finite element formulation. Using a standard finite element technique, the resulting algebraic equations can be written in matrix form for each element, producing a set of coupled matrix equations. These equations can then be assembled, producing one large matrix equation where it can be seen that the final coefficient matrix is simply a reorganization of the individual coefficient matrices; that is, unlike the displacement formulation, this method requires no addition during assembly.

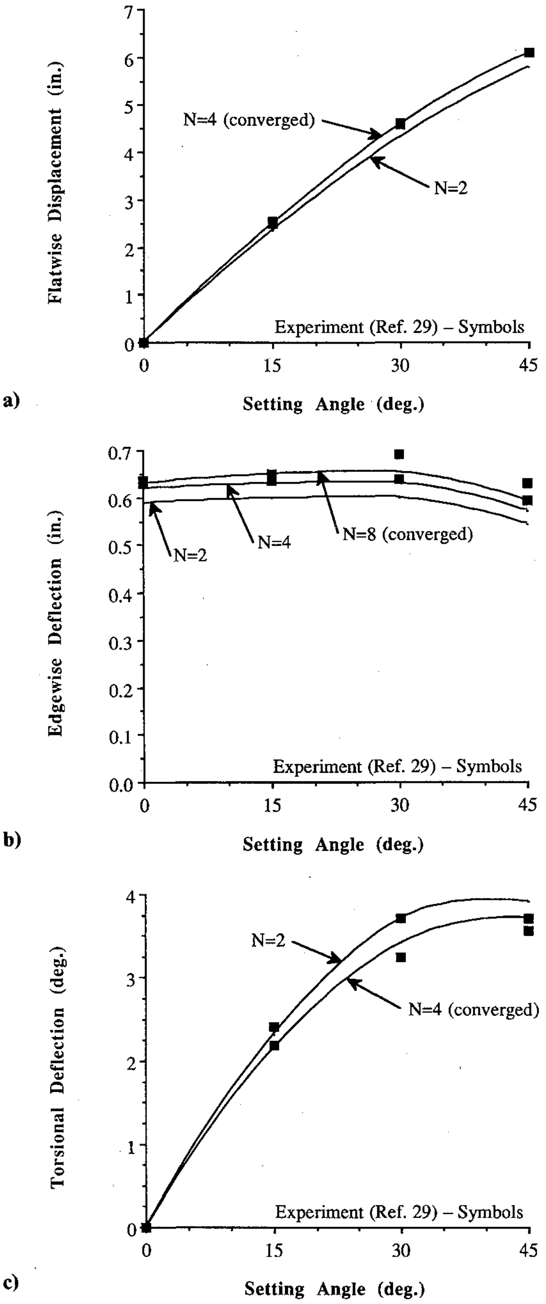


**Fig. 3 Tip deflections of the Princeton beam for different setting angles.**

**Table 1 Stiffnesses for solid beam from NABSA**

| Stiffness | NABSA |
|---|---|
| $A_{11}$, lb | $0.8060 \times 10^6$ |
| $A_{12}$, lb | $-0.4623 \times 10^5$ |
| $A_{22}$, lb | $0.9304 \times 10^5$ |
| $A_{33}$, lb | $0.6685 \times 10^4$ |
| $D_{11}$, lb-in.$^2$ | $0.1223 \times 10^3$ |
| $D_{12}$, lb-in.$^2$ | $0.3376 \times 10^2$ |
| $D_{22}$, lb-in.$^2$ | $0.1816 \times 10^3$ |
| $D_{33}$, lb-in.$^2$ | $0.9116 \times 10^5$ |

The perturbed state of the weakest possible form, the Jacobian of $\overline{\delta\pi}$, can be found from Eq. (45). After that, it is easy to solve the resulting nonlinear equations by a Newton-Raphson method. Plugging in the shape functions described and carrying out integration from 0 to $\ell$, one can obtain the algebraic form of $\overline{\delta\pi}$ and its Jacobian $\mathcal{K}$. Then, in accordance with the usual Newton-Raphson method, the column matrix of unknowns $v$ can be updated at the $k$th iteration

$$v_{k+1} = v_k + \Delta v_k \tag{51a}$$

$$\mathcal{K}_k \Delta v_k = \mathcal{F}_k \tag{51b}$$

where $\mathcal{K}_k$ is the Jacobian matrix for the $k$th iteration and $\mathcal{F}_k$ is the residual load column matrix for the $k$th iteration. There will be a total of $18N + 24$ unknowns including 24 hatted quantitites at the boundaries and $18N + 12$ equations. However, 12 of the 24 hatted quantities will be known quantities. For the clamped-free case, for instance, three displacements and three rotation variables will be zero at the clamped end, and three force and three moment components will be zero at the free end. Then displacements and rotations are unknown at the free end, whereas forces and moments are unknown and at the clamped end.

## Applications

Results from our nonlinear, one-dimensional, global deformation analysis have been compared with those from an experiment involving an aluminum beam, with a solid rectangular cross section, undergoing large deformations.[35] In the experiment, the beam was mounted so that, when undeformed, its longitudinal axis was in the horizontal plane. A weight several times that of the beam was mounted at the tip, and the deflections and torsional rotation were measured for various values of an angle (called herein the setting angle) that measured the orientation of the undeformed beam's major cross-sectional principal axes relative to the vertical plane. The properties used in the present calculation are essentially those of the EBT St. Venant solutions and are given in Ref. 36. The correlations are presented in Fig. 3 for a tip weight of 3 lb; for this case the deflections reach 35% of the beam length. Excellent correlation is obtained—even for a very small number of elements.

Next we correlate our unified analysis with results obtained from a large-deflection experiment using a composite beam.[37] This beam was made of AS4/3501-6 Graphite/Epoxy with layup [45 deg/0 deg]$_{3s}$ having length 22 in.; the rectangular cross section had horizontal and vertical dimensions 1.18 in. and 0.0575 in., respectively. In this application, the cantilevered beam was subjected to applied loading at the tip in the $x_1$-$x_3$ plane. The stiffnesses determined by NABSA are presented in Table 1. (Note that we are treating the cross section of this "strip" as a solid one; otherwise we would have to consider the influence of restrained warping, which is beyond the scope of the present work.)

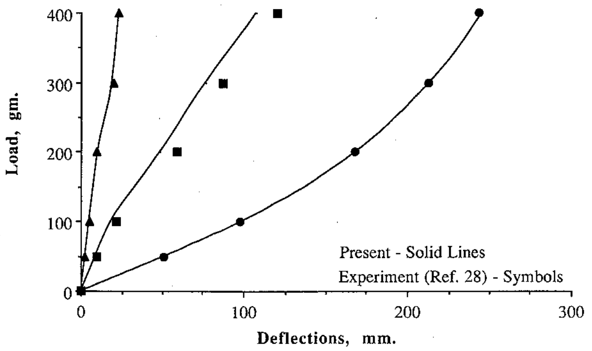


**Fig. 4 Tip deflections of the MIT beam for different load levels.**

Notice the elastic coupling is exhibited in the $A$ and $D$ matrices; $A_{12}$ is extension-shear coupling (i.e., horizontal transverse shear), and $D_{12}$ is bending-torsion coupling (i.e., vertical bending). Extension-shear coupling is shown to be a concomitant coupling in closed-cell, thin-walled beams with bending-torsion coupling by Rehfield and Atılgan in Refs. 10 and 38. Therein, they also show that bending-shear coupling accompanies extension-twist coupling in such beams (through nonzero terms in the $B$ submatrix). Effects of these concomitant couplings, especially the latter one, can be extremely important in thin-walled, closed-cell beams, as shown in Ref. 39, and certain of those effects are independent of slenderness.

The measured tip displacements from Ref. 37 and the present results are shown in Fig. 4. Excellent correlation is obtained. As a result of the bending-torsion coupling and its accompanying extension-shear coupling created by the selection of this particular layup, a significant $u_2$ deflection is obtained. This is in the direction normal to the plane containing the load vector and the undeformed beam and takes place by means of the following mechanisms. The beam is bent about the $x_2$ axis due to the loading in the $x_3$ direction; this means that a typical cross-sectional frame rotates about the $x_2$ axis. At the same time, because of the bending-torsion coupling ($D_{12} \neq 0$), the beam is forced to twist; that is, a typical cross-sectional frame rotates about the $x_1$ axis. The twisting rotation about the $x_1$ axis means that there is now a component of shear force in the local $x_2$ direction leading to a shear strain $\gamma_{12}$ and a $u_2$ displacement. Due to the concomitant extension-shear coupling ($A_{12} \neq 0$), the shear force in the $x_2$ direction also induces a very small axial displacement in addition to that already taking place due to pure geometrical nonlinearity (the so-called foreshortening effect).

## Concluding Remarks

A unified methodology for geometrically nonlinear analysis of nonhomogeneous, anisotropic beams is presented. A two-dimensional cross-sectional analysis and a nonlinear one-dimensional global deformation analysis are derived from the common framework of a three-dimensional, geometrically nonlinear theory of elasticity. The only restrictions are that the strain and local rotation are small compared to unity and that warping displacements are small relative to the cross-sectional dimensions. Here, warping refers to all possible deformations of the beam cross section, both in and out of the cross-sectional plane. The present equations are identical in form to those of Ref. 20, except that the linear global strain variables of Ref. 20 are replaced in the present analysis by the nonlinear ones, and the linear equilibrium equations are replaced by the exact intrinsic, nonlinear equilibrium equations.

The two-dimensional, cross-sectional analysis is valid for beams having cross sections with arbitrary shapes and made out of nonhomogeneous and anisotropic materials if typical open cross-section effects, such as restrained warping and the trapeze effect, are unimportant. Solutions of the equations governing the cross-sectional problem are discussed for various special cases. Also, approximations that are necessary to solve the equations for both extension-bending-torsion problems and the flexure problem are presented. Such approximations include linearizing the global equilibrium equations about the reference state or applying asymptotical methods to obtain a sectional stiffness matrix.

The present one-dimensional global deformation analysis turns out to be equivalent to the static intrinsic equations as derived in Ref. 25 from equilibrium considerations and in Ref. 23 by energy methods. The weakest possible form of all the governing equations is presented and used as a basis for a nonlinear, mixed, finite element, global-deformation analysis.

Thus, the unified methodology yields a two-stage analysis that exhibits excellent correlation with published experimental results for both isotropic and composite beams. Practically speaking, this suggests that the section properties obtained from a linear, two-dimensional, finite element analysis based on Ref. 20 can be used with confidence in nonlinear analysis of composite beams. This confidence is well founded as long as 1) the global deformation analysis is formulated in terms of the present global strain measures $\gamma$ and $\kappa$; 2) the elastic constants are incorporated in a linear relation between section forces and moments $F$ and $M$, expressed in the deformed beam basis, and $\gamma$ and $\kappa$; and 3) the deformations are not large enough to alter the elastic constants.

Concerning future work, an extension of the present methodology to treat initially curved and twisted beams would be a straightforward exercise. Furthermore, the asymptotical accuracy of results from NABSA needs to be verified. Additional development of the unified methodology is necessary to treat the effects of warping on the virtual work of applied loads and body forces, deformations that are large enough to significantly alter the elastic constants during deformation, and beams with open cross sections, in which both trapeze effects and restrained warping effects need to be included. To treat these phenomena may require additional displacement variables in the global deformation analysis and an extension of the present analysis to account, at least approximately, for moderate local rotation as described in Refs. 21 and 22. Also, for applications involving optimization, and particularly as additional geometric and physical effects are considered, improving computational efficiency of the sectional analysis becomes more critical.

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